2.2b nonlinear difference equations

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Def 2.1 For the 1st difference equation / 1st-order system

$$x_{t+1} = f(x_t), \qquad X(t+1) = F(X(t))$$

an **equilibrium solution** or **steady-state solution** is a constant solution $\bar{x}$ ($\bar{X}$) to the difference equation. i.e.

$$\bar{x} = f(\bar{x}) \qquad / \qquad \bar{X} = F(\bar{X})$$

$\bar{x}$ and $\bar{X}$ are **fixed pts** of respectively f or F.

Notation: Let $f^t(x_0) = \underbrace{f \circ f \circ \cdots \circ f}_{t \text{ times}}(x_0)$. So, if $x_{t+1} = f(x_t)$, then $f^t(x_0) = x_t$

Ex. Let $x_{t+1} = \frac{x_t}{2} + 5$. Then $\bar{x} = 10$ is a steady-state sol.

$\uparrow = f(x_t) = \frac{x_t}{2} + 5$

Ex. Let $X(t+1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X(t)$. Then $\bar{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a steady-state sol.

Def. 2.2 A **periodic solution of period $m > 1$** of a difference eq $x_{t+1} = f(x_t)$ is a real-valued sol $\bar{x}_k$ satisfying

$$f^m(\bar{x}_k) = \bar{x}_k \quad \text{and} \quad f^i(\bar{x}_k) \neq \bar{x}_k \quad \text{for } i = 1, \dots, m-1$$

An **m-cycle** is a set of pts $\{\bar{x}_1, \dots, \bar{x}_m\}$ where $f(\bar{x}_k) = \bar{x}_{k+1}$ and each pt $\bar{x}_k$ for $k = 1, \dots, m$ is a periodic solution of period m.

The set $\{\bar{x}_1, f(\bar{x}_1), \dots, f^{m-1}(\bar{x}_1)\}$ is the **periodic orbit of $\bar{x}_1$**.

Similar definitions for a first-order system $X(t+1) = F(X(t))$

Aside: If $\bar{x}_k$ is a periodic solution to $x_{t+1} = f(x_t)$ of period m, Aside: If $\bar{x}_k$ is a periodic solution to $x_{t+1}=f(x_t)$ of period m, then $\bar{x}_k$ is a fixed pt of $f^m, f^{2m}, f^{3m}, \ldots$

Aside: By def., a solution of period m can't have period $k<m$.

Ex. Let $x_{t+1} = f(x_t)$, where $f(x) = -x$

Then $\bar{x} \in \mathbb{R}$ for any $\bar{x} \neq 0$ is a periodic solution of period 2.

Suppose $\bar{x}=0$. Then $f(0)=0$, so this a steady state equilibrium instead.

Ex. let $x_{t+1} = \dfrac{a x_t}{b + x_t} = f(x_t)$, $a, b > 0$.

To solve for an equilibrium solution, solve $\bar{x} = \dfrac{a\bar{x}}{b+\bar{x}}$

$$\Rightarrow \bar{x}(b+\bar{x}) = a\bar{x}$$
$$\bar{x}^2 + b\bar{x} - a\bar{x} = 0$$
$$\bar{x}(\bar{x} + b - a) = 0$$
$$\Rightarrow \underbrace{\bar{x} = 0, \quad a-b}_{\text{equilibrium solutions}}$$

Are there any 2-cycles? Solve for $f^2(\bar{x}) = f(f(\bar{x})) = \bar{x}$

$$\Rightarrow f\left(\frac{a\bar{x}}{b+\bar{x}}\right) = \bar{x}$$

$$\Rightarrow \frac{a\left(\frac{a\bar{x}}{b+\bar{x}}\right)}{b+\left(\frac{a\bar{x}}{b+\bar{x}}\right)} = \bar{x} \quad \Rightarrow a\left(\frac{a\bar{x}}{b+\bar{x}}\right) = b\bar{x} + \bar{x}\left(\frac{a\bar{x}}{b+\bar{x}}\right)$$

$$\Rightarrow a^2\bar{x} = b^2\bar{x} + b\bar{x}^2 + a\bar{x}^2$$
$$\Rightarrow (a^2-b^2)\bar{x} - (a+b)\bar{x}^2 = 0$$
$$(a+b)\bar{x}(a-b-\bar{x}) = 0$$
$$\Rightarrow \bar{x} = 0 \text{ or } \bar{x} = a-b$$

But both of these actually have period 1 because they are equilibria that we found earlier, so there are no 2-cycles.

Def. 2.3a An equilibrium solution $\bar{x}$ of $x_{t+1} = f(x_t)$ is **locally stable** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x_0 - \bar{x}| < \delta$, then

$$|x_t - \bar{x}| = |f^t(x_0) - \bar{x}| < \varepsilon \quad \forall t \geq 0.$$

If $\bar{x}$ is not stable, then it is **unstable.**

Ex. $x_{t+1} = \frac{1}{2} x_t$ has a locally stable equilibrium solution $\bar{x} = 0$

Ex. $x_{t+1} = 2x_t$ has a locally unstable equilibrium solution $\bar{x} = 0$

Def. 2.3b An equilibrium solution $\bar{x}$ of $x_{t+1} = f(x_t)$ is **locally attracting** if $\exists \gamma > 0$ s.t. for all x_0 s.t. $|x_0 - \bar{x}| < \gamma$,

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} f^t(x_0) = \bar{x}$$

Ex. $x_{t+1} = \frac{1}{2} x_t$ has a locally attracting sol $\bar{x} = 0$.

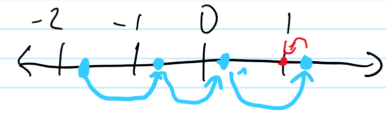
Ex. $x_{t+1} = 2x_t$. $\bar{x} = 0$ is not a locally attracting sol.

Def. 2.3c The equilibrium solution $\hat{x}$ is **locally asymptotically stable** if it is both locally attracting and locally stable.

Ex. $x_{t+1} = \frac{1}{2} x_t$ has a locally stable attracting solution $\bar{x} = 0$.

Important: It is possible to be locally attracting but not locally stable.

Ex.
$$f(x) = \begin{cases} x+1, & \text{if } x<1 \\ 1, & \text{if } x \geq 1 \end{cases}$$



$x_{t+1} = f(x_t)$ has an equilibrium solution at $\bar{x} = 1$

And it is locally attracting because $\lim_{t\to\infty} x_t = 1$ for all initial conditions.

But let $\varepsilon = \frac{1}{2}$. Then for $\frac{1}{2} < x_0 < 1$, $x_1 > \frac{3}{2}$, so $|x_1 - 1| > \frac{1}{2}$, so there is no $\delta > 0$ that works.

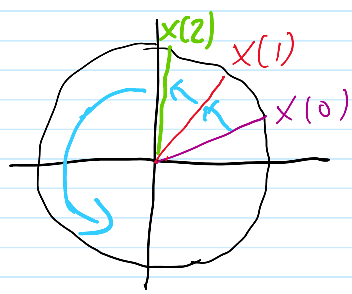
Important: It is possible to be locally stable but not locally attracting

Ex. Let
$$X(t+1) = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation matrix}} X(t),$$
where θ is the angle of rotation

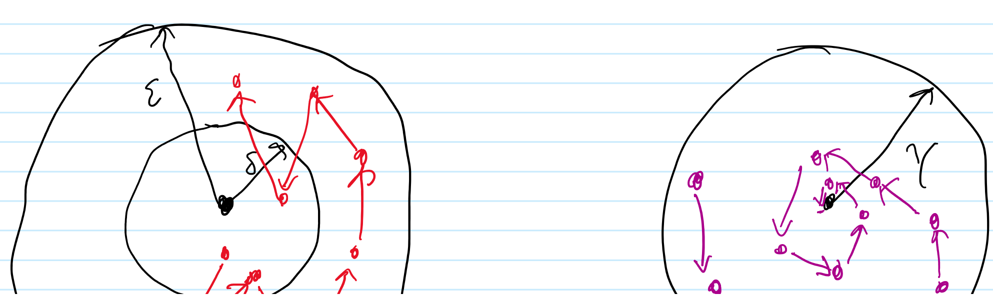
Then $\bar{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a locally stable equilibrium because if $\|X(0)\|_2 < r$, then $\|X(0) - \bar{X}\|_2 < r$

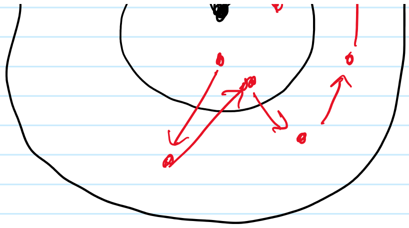
But for any $X(0) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, it will never converge to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus, it is not locally attracting.



Intuition:





Stable means that if a vector starts within distance δ, it stays within dist ε

Attracting means that in the limit, if a trajectory starts within dist γ, then it converges to the equilibrium.